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DERIVATION OF RELIABILITY FORMULAS FOR A THREE-DIMENSIONAL MATRIX OF EQUIVALENT COMPONENTS

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DERIVATION OF RELIABILITY FORMULAS FOR A THREE-DIMENSIONAL MATRIX OF EQUIVALENT COMPONENTS

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APPROVED:

W. L. Finch

W.L.FINCH, MANAGER
PROGRAM 461 PRODUCT ASSURANCE

APPROVED:

J. C. Solvason

J.C.SOLVASON
461 PROGRAM MANAGER

FOREWORD

This report presents original work by Dr. A. V. Pershing on determining the reliability of three-dimensional matrices. The work was accomplished under Air Force Contract AF 04(647)-787.

ABSTRACT

This report develops a set of equations that describe the reliability of any three-dimensional matrix of equivalent components. In addition, it gives the step-by-step mathematical sequence used in deriving the general equations from the equations for the most simple two-dimensional parallel-series arrangement of components.

CONTENTS

CONTENTS (Continued)

ILLUSTRATIONS

NOTATION

f	component failure factor
ℓ	limit for number of components on the x-axis
m	limit for number of components on the y-axis
n	limit for number of components on the z-axis
p	component reliability factor (equivalent to r)
q	component reliability factor (equivalent to f)
R_I	signal path
R_k	reliability of any element
R_t	total reliability
r	component reliability factor
r_t	reliability of a component with respect to time
r_L	lower limit for summation
r_U	upper limit for summation
s'	signal source
s''	signal destination
u	$\ell \cdot m$
λ	component failure constant

Section 1
INTRODUCTION

In the design of modern electronic equipment, the ability to establish and predict the reliability of complex groups of components has become increasingly important. Today, designers are asked frequently to determine the reliability of component groups that may contain thousands of components connected in multiple parallel-series arrangements. In these arrangements, various combinations of failed and successful components may be allowed, and the components may be placed in either two or three dimensions. These conditions result in highly complex patterns of successful signal paths, and the methods used previously to determine the reliability of simpler parallel and series arrangements are no longer adequate.

As a result, new methods had to be devised to accurately describe the reliability of these complex component groups. The initial work in this area – the algebra for determining the probability of signal success where signal paths are not mutually exclusive – was done by Harold Balaban and others.* This work laid the mathematical foundation which led to practical formulas that approximated the reliability of multiple parallel-series arrangements of components. In this report, formulas are developed that will accurately describe the reliability of any two- or three-dimensional group of equivalent components.

Matrices are used in this report as mathematical models for complex component groups, and a unique method is employed in developing the equations describing the reliability of these matrices. First, equations are derived empirically to given the reliability of any two-dimensional matrix of equivalent components. Then, these equations are expanded in steps to describe any three-dimensional matrix of equivalent components.

*ARINC Research Corporation, The Effect of Redundancy on System Reliability, by Harold Balaban, Washington, D.C., May 1959.

Finally, the binomial theorem is applied to the resulting equations. This application makes it possible to readily determine the reliability of any matrix with any combination of failed and successful components.

Section 2
GENERAL SOLUTION FOR SIGNAL SUCCESS IN A TWO-DIMENSIONAL MATRIX

2.1 GENERAL

In this section, a formula is developed empirically for describing the reliability (probability of signal success) of any two-dimensional matrix with the following characteristics:

- Matrix contains any number of sets connected in series
- Sets contain any number of members (components) connected in parallel
- Components have the same characteristic failure rate

Derivation of the formula begins with developing a formula for the simplest matrix containing series sets with parallel components – two sets with two components in each set. Then, the resultant formula is expanded in steps to include all two-dimensional matrices with these characteristics.

2.2 SERIES OF TWO SETS, EACH SET CONTAINING TWO PARALLEL COMPONENTS

First, consider the arrangment of the four components shown in Fig. 2-1. There are two sets (r_1 and r_2) connected in series; each set contains two parallel-connected members. The probability of a signal traveling from S' to S'' is dependent on the possible paths that the signal can take and on the path overlaps. Figure 2-2 shows the four possible signal paths (R_I, R_{II}, R_{III}, and R_{IV}).

The total probability or reliability (R_t) of a signal traveling from S' to S'' is found by summing each separate signal path (R_I, R_{II}, R_{III}, and R_{IV}), subtracting the first overlap (sum of products $R_I \cdot R_{II}$, $R_I \cdot R_{III}$, etc.), adding the second overlap ($R_I \cdot R_{II} \cdot R_{III}$, $R_I \cdot R_{II} \cdot R_{IV}$, etc.), and subtracting the third overlap ($R_I \cdot R_{II} \cdot R_{III} \cdot R_{IV}$).

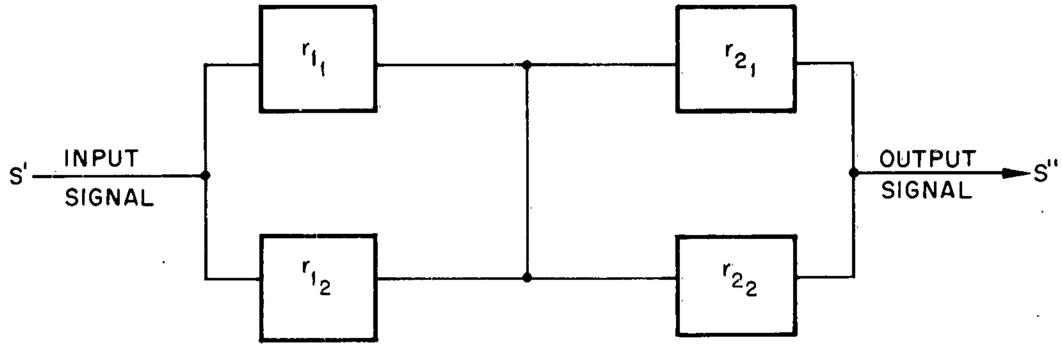

Fig. 2-1 Series of Two Sets, Each Containing Two Parallel Components

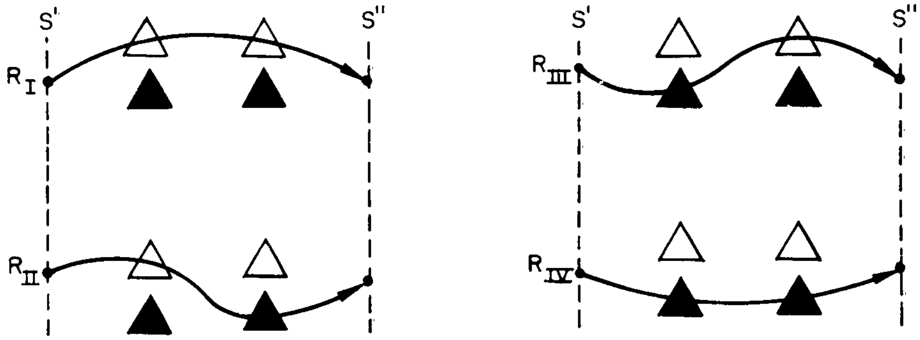

Fig. 2-2 Possible Signal Paths in a Series of Two Sets, Containing Two Parallel Components

Then

$$R_t = \left[R_I + R_{II} + R_{III} + R_{IV}\right] - \left[R_I \cdot R_{II} + R_I \cdot R_{III} + R_I \cdot R_{IV} + R_{II} \cdot R_{III} + R_{II} \cdot R_{IV} + R_{III} \cdot R_{IV}\right] + \left[R_I \cdot R_{II} \cdot R_{III} + R_I \cdot R_{II} \cdot R_{IV} + R_I \cdot R_{III} \cdot R_{IV} + R_{II} \cdot R_{III} \cdot R_{IV}\right] - \left[R_I \cdot R_{II} \cdot R_{III} \cdot R_{IV}\right] \qquad (2.1)$$

By substituting the component designations given in Fig. 2-1, Eq. (2.1) becomes

$$
\begin{aligned}
R_t = {} & \left[r_{1_1}r_{2_1} + r_{1_1}r_{2_2} + r_{1_2}r_{2_1} + r_{1_2}r_{2_2}\right] - \Big[\left(r_{1_1}r_{2_1}\right)\left(r_{1_1}r_{2_2}\right) + \left(r_{1_1}r_{2_1}\right)\left(r_{1_2}r_{2_1}\right) \\
& + \left(r_{1_1}r_{2_1}\right)\left(r_{1_2}r_{2_2}\right) + \left(r_{1_1}r_{2_2}\right)\left(r_{1_2}r_{2_1}\right) + \left(r_{1_1}r_{2_2}\right)\left(r_{1_2}r_{2_2}\right) \\
& + \left(r_{1_2}r_{2_1}\right)\left(r_{1_2}r_{2_2}\right)\Big] + \Big[\left(r_{1_1}r_{2_1}\right)\left(r_{1_1}r_{2_2}\right)\left(r_{1_2}r_{2_1}\right) + \left(r_{1_1}r_{2_1}\right)\left(r_{1_1}r_{2_2}\right)\left(r_{1_2}r_{2_2}\right) \\
& + \left(r_{1_1}r_{2_1}\right)\left(r_{1_2}r_{2_1}\right)\left(r_{1_2}r_{2_2}\right) + \left(r_{1_1}r_{2_1}\right)\left(r_{1_2}r_{2_1}\right)\left(r_{1_2}r_{2_2}\right)\Big] \\
& - \Big[\left(r_{1_1}r_{2_1}\right)\left(r_{1_1}r_{2_2}\right)\left(r_{1_2}r_{2_1}\right)\left(r_{1_2}r_{2_2}\right)\Big]
\end{aligned}
\qquad (2.2)
$$

By striking component duplications in each overlap term and by setting <u>all</u> r's equal (same failure constant), Eq. (2.2) becomes

$$
\begin{aligned}
R_t &= \left[4r^2\right] - \left[4r^3 + 2r^4\right] + \left[4r^4\right] - \left[r^4\right] \\
&= 4r^2 - 4r^3 + r^4 \\
&= r^2\left[4(1 - r) + r^2\right] \\
&= \left[1 - (1 - r)^2\right]^2 \quad \text{or} \quad \left[1 - f^2\right]^2
\end{aligned}
\qquad (2.3)
$$

Thus, Eq. (2.3) describes the reliability of the series of sets shown in Fig. 2-1.

Equation (2.3) can be generalized as follows:

$$R_t = R^2 = \Pi_k R_k = \Pi_k \left[1 - f_i \cdot f_j\right]_k$$

or

$$R_t = \Pi_k \left[1 - \Pi_j f_j\right]_k \tag{2.4}$$

where

k = any set

j = any component

2.3 SERIES OF THREE SETS, EACH SET CONTAINING TWO PARALLEL COMPONENTS

The effect of adding another like set to the series shown in Fig. 2-1 can be determined by using the procedure described in paragraph 2.2. Figure 2-3 illustrates the new series.

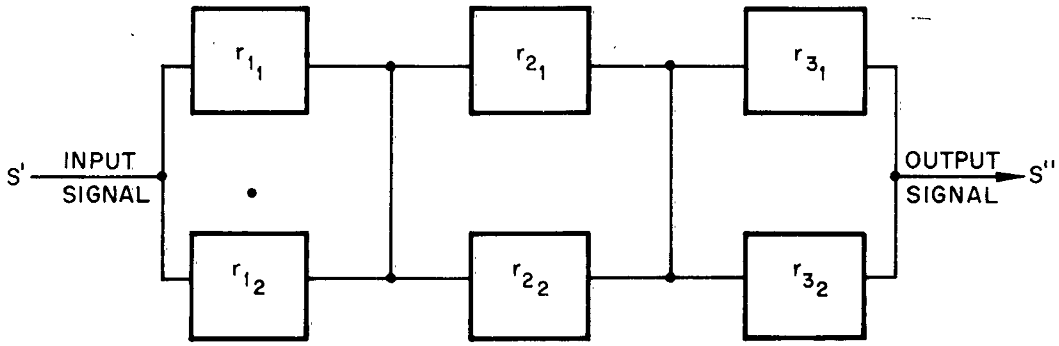

Fig. 2-3 Series of Three Sets, Each Containing Two Parallel Components

By constructing a diagram similar to the one shown in Fig. 2-2, it can be seen that there are eight possible signal paths (R_I through R_{VIII}). Hence, the reliability for this series can be expressed as

$$
\begin{aligned}
R_t = & \left[R_I + R_{II} + R_{III} + R_{IV} + R_V + R_{VI} + R_{VII} + R_{VIII}\right] \quad \text{(8 terms)} \qquad (2.5) \\
& - \left[R_I \cdot R_{II} + R_I \cdot R_{III} + \ldots + R_{VI} \cdot R_{VIII} + R_{VII} \cdot R_{VIII}\right] \quad \text{(28 terms)} \\
& + \left[R_I \cdot R_{II} \cdot R_{III} + \ldots + R_{VI} \cdot R_{VII} \cdot R_{VIII}\right] \quad \text{(56 terms)} \\
& - \left[R_I \cdot R_{II} \cdot R_{III} \cdot R_{IV} + \ldots + R_V \cdot R_{VI} \cdot R_{VII} \cdot R_{VIII}\right] \quad \text{(70 terms)} \\
& + \left[(R_I \cdot \ldots \cdot R_V) + \ldots + (R_{IV} \cdot \ldots \cdot R_{VIII})\right] \quad \text{(56 terms)} \\
& - \left[(R_I \cdot \ldots \cdot R_{VI}) + \ldots + (R_{III} \cdot \ldots \cdot R_{VIII})\right] \quad \text{(28 terms)} \\
& + \left[(R_I \cdot \ldots \cdot R_{VII}) + \ldots + (R_{II} \cdot \ldots \cdot R_{VIII})\right] \quad \text{(8 terms)} \\
& - \left[R_I \cdot \ldots \cdot R_{VIII}\right] \quad \text{(1 term)}
\end{aligned}
$$

By substituting the designations given in Fig. 2-3, Eq. (2.5) becomes

$$
\begin{aligned}
R_t = & \left[\left(r_{1_1} r_{2_1} r_{3_1}\right) + \left(r_{1_1} r_{2_1} r_{3_2}\right) + \left(r_{1_1} r_{2_2} r_{3_1}\right) + \left(r_{1_1} r_{2_2} r_{3_2}\right)\right. \qquad (2.6) \\
& \left. + \left(r_{1_2} r_{2_1} r_{3_1}\right) + \left(r_{1_2} r_{2_1} r_{3_2}\right) + \left(r_{1_2} r_{2_2} r_{3_1}\right) + \left(r_{1_2} r_{2_2} r_{3_2}\right)\right] \\
& - \left[\left(r_{1_1} r_{2_1} r_{3_1}\right)\left(r_{1_1} r_{2_1} r_{3_2}\right) + \ldots + \left(r_{1_2} r_{2_2} r_{3_1}\right)\left(r_{1_2} r_{2_2} r_{3_2}\right)\right] \quad \text{(28 terms)}
\end{aligned}
$$

$$+\left[\left(r_{1_1} r_{2_1} r_{3_1}\right)\left(r_{1_1} r_{2_1} r_{3_2}\right)\left(r_{1_1} r_{2_2} r_{3_1}\right) + \cdots \right.$$

$$\left. + \left(r_{1_2} r_{2_1} r_{3_2}\right)\left(r_{1_2} r_{2_2} r_{3_1}\right)\left(r_{1_2} r_{2_2} r_{3_2}\right)\right] \quad \text{(56 terms)}$$

$$-\left[\left(r_{1_1} r_{2_1} r_{3_1}\right)\left(r_{1_1} r_{2_1} r_{3_2}\right)\left(r_{1_1} r_{2_2} r_{3_1}\right)\left(r_{1_1} r_{2_2} r_{3_2}\right) + \cdots \right.$$

$$\left. + \left(r_{1_2} r_{2_1} r_{3_1}\right)\left(r_{1_1} r_{2_1} r_{3_2}\right)\left(r_{1_1} r_{2_2} r_{3_1}\right)\left(r_{1_2} r_{2_2} r_{3_2}\right)\right] \quad \text{(70 terms)}$$

$$+\left[\left(r_{1_1} r_{2_1} r_{3_1}\right)\left(r_{1_1} r_{2_1} r_{3_2}\right)\left(r_{1_1} r_{2_2} r_{3_1}\right)\left(r_{1_1} r_{2_2} r_{3_2}\right)\left(r_{1_2} r_{2_1} r_{3_1}\right) + \cdots \right.$$

$$\left. + \left(r_{1_1} r_{2_2} r_{3_3}\right)\left(r_{1_2} r_{2_1} r_{3_1}\right)\left(r_{1_2} r_{2_1} r_{3_2}\right)\left(r_{1_2} r_{2_2} r_{3_1}\right)\left(r_{1_2} r_{2_2} r_{3_2}\right)\right] \quad \text{(56 terms)}$$

$$-\left[\left(r_{1_1} r_{2_1} r_{3_1}\right) \cdot \ldots \cdot \left(r_{1_2} r_{2_1} r_{3_2}\right) + \cdots \right.$$

$$\left. + \left(r_{1_1} r_{2_2} r_{3_1}\right) \cdot \ldots \cdot \left(r_{1_2} r_{2_2} r_{3_2}\right)\right] \quad \text{(28 terms)}$$

$$+\left[\left(r_{1_1} r_{2_1} r_{3_1}\right) \cdot \ldots \cdot \left(r_{1_2} r_{2_2} r_{3_1}\right) + \cdots \right.$$

$$\left. + \left(r_{1_1} r_{2_1} r_{3_2}\right) \cdot \ldots \circ \left(r_{1_2} r_{2_2} r_{3_2}\right)\right] \quad \text{(8 terms)}$$

$$-\left[\left(r_{1_1} r_{2_1} r_{3_1}\right) \cdot \ldots \circ \left(r_{1_2} r_{2_2} r_{3_2}\right)\right] \quad \text{(1 term)}$$

By striking out component duplications in each overlap term and by setting all r's equal (same failure constant), Eq. (2.6) becomes

$$
\begin{aligned}
R_t &= 0 \cdot r^0 + 0 \cdot r^1 + 0 \cdot r^2 + 8 \cdot r^3 - 12 \cdot r^4 + 6 \cdot r^5 - r^6 \\
&= \left[\left(2r - r^2\right)^3\right] \\
&= \left[1 - (1 - r)^2\right]^3 \quad \text{or} \quad \left[1 - f^2\right]^3 \qquad (2.7)
\end{aligned}
$$

Thus, Eq. (2.7) describes the reliability of the series of sets shown in Fig. 2-3.

Equation (2.7) can be generalized as follows:

$$R_t = R^3 = \Pi_k R_k = \Pi_k \left[1 - f_i \cdot f_j\right]_k$$

or

$$R_t = \Pi_k \left[1 - \Pi_j f_j\right]_k \qquad (2.8)$$

where

k = any set

j = any component

2.4 SERIES OF ANY NUMBER OF SETS, EACH SET CONTAINING TWO PARALLEL COMPONENTS

By using the method employed in paragraphs 2.2 and 2.3, the reliability formula can be determined for a series of any number of sets that contain two parallel components.

The formulas describing these series follow a definite pattern as indicated in the tabulation below:

No. of Sets	Formula
1	$R_t = 1 - (1 - r)^2$
2	$= \left[1 - (1 - r)^2\right]^2$
3	$= \left[1 - (1 - r)^2\right]^3$
4	$= \left[1 - (1 - r)^2\right]^4$
.	.
.	.
.	.
n	$R_t = \left[1 - (1 - r)^2\right]^n$

Hence, the reliability for any series of this type can be described by

$$R_t = \left[1 - (1 - r)^2\right]^n \qquad (2.9)$$

The general formula for this series can be expressed as

$$R_t = \prod_{1k}^{n} \left[1 - \prod_{1j}^{m} \left(1 - r_j\right)_j\right]_k \qquad (2.10)$$

2.5 SERIES OF n SETS, EACH SET CONTAINING m PARALLEL-CONNECTED COMPONENTS

Equation (2.9) describes the reliability of a series containing any number of sets, but each set is restricted to two parallel-connected members. In order to describe the

reliability of any two-dimensional matrix of this type, it is necessary to remove this restriction and to extend the members of the sets to any number as shown in Fig. 2-4.

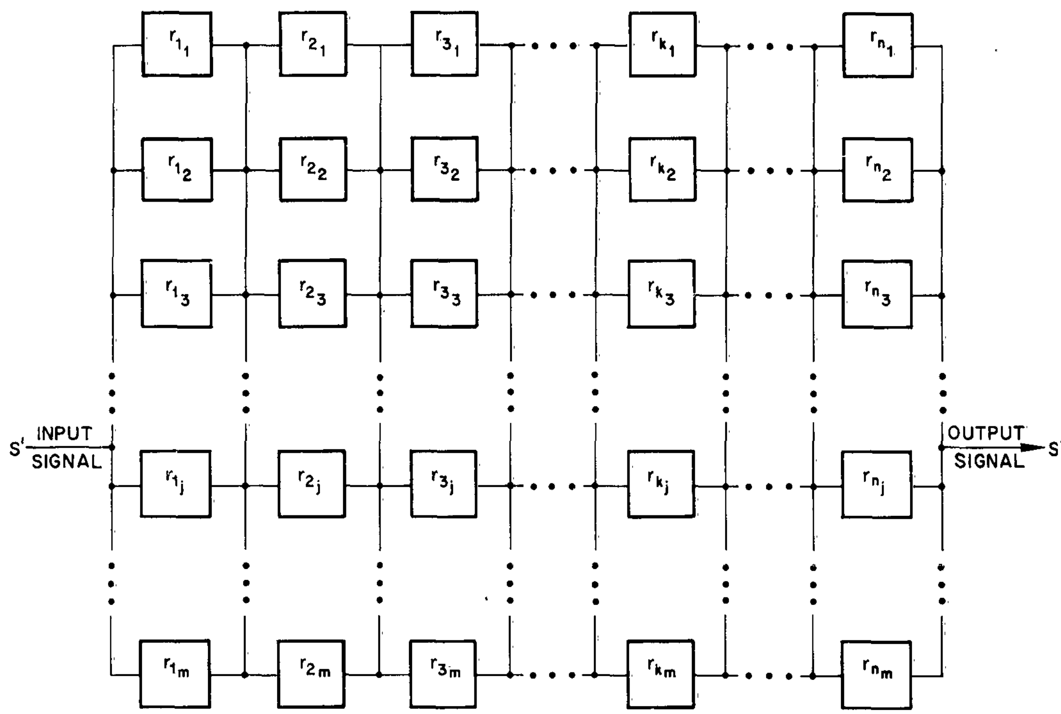

Fig. 2-4 Series of n Sets, Each Containing m Parallel-Connected Components

By extrapolation from Eq. (2.10), the number of parallel-connected components in each set can be extended as follows:

$$R_t = \prod_{1k}^{n} \left[1 - \prod_{1j}^{m} \left(1 - r_j\right)_j\right]_k$$

If for any value of k

$$r_{1_k} = r_{2_k} = r_{3_k} = r_{j_k} = r_k \quad \text{(constant)}$$

then

$$R_t = \prod_{1_k}^{n} \left[1 - \left(1 - r_k\right)^m\right]_k$$

and if for any values of j and k

$$r_{j_k} = r_{1_1} = r_{2_1} = r_{j_1} = r \quad \text{(constant)}$$

then

$$R_t = \left[1 - (1 - r)^m\right]^n \tag{2.11}$$

where

m = number of parallel components in each set
n = number of sets in series

Thus, Eq. (2.11) describes the reliability of any two-dimensional matrix that contains a series of sets in which the members of the sets are equivalent and connected in parallel.

Section 3
GENERAL SOLUTION FOR SIGNAL SUCCESS IN A THREE-DIMENSIONAL MATRIX

3.1 GENERAL

In this section, a formula is derived for describing the reliability of any three-dimensional matrix with the following characteristics:

- Matrix contains any number of elements connected in series
- Elements contain any number of components connected in parallel
- Any element may have any characteristic failure rate

Derivation of the formula begins by converting the formula for the general two-dimensional matrix into a formula for a three-dimensional matrix. Then, the resultant formula is expanded in steps to include all three-dimensional matrices with these characteristics.

3.2 CONVERSION OF THE GENERAL TWO-DIMENSIONAL MATRIX INTO A THREE-DIMENSIONAL MATRIX

A general two-dimensional matrix is shown in Fig. 2-4. This matrix consists of a series of n sets where each set contains a total of m components connected in parallel.

Now, if the matrix is folded on itself as shown in Fig. 3-1, a three-dimensional matrix is formed. Since the connections between the components have not been changed by folding, everything established empirically in Section 2 for the two-dimensional matrix is applicable also to this three-dimensional matrix.

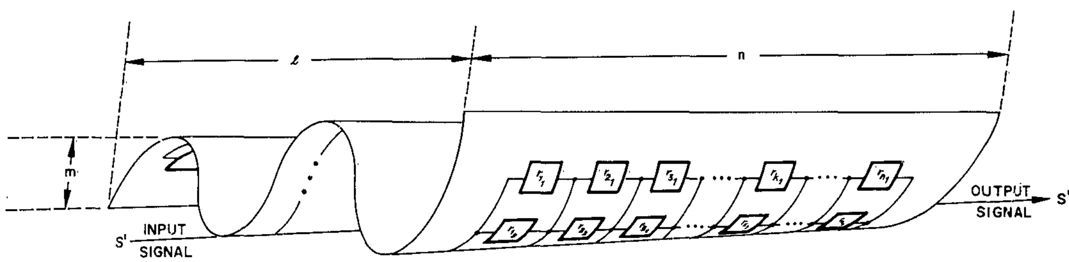

Fig. 3-1 Forming a Three-Dimensional Matrix from a Two-Dimensional Matrix

Since another dimension (ℓ) has been added by folding, the total number of parallel components in each set is determined by the product of the number of sheets (ℓ) formed by the folding and the number of components (m) on each sheet.

If each set of parallel components is considered an element, the matrix shown in Fig. 3-1 can be conceived as a group of series-connected elements like those shown in Fig. 3-2.

3.3 RELIABILITY OF A THREE-DIMENSIONAL MATRIX COMPOSED OF EQUIVALENT ELEMENTS

Now, since the three-dimensional matrix was formed from the two-dimensional matrix without altering the connections between components, the two-dimensional matrix equation (2.11),

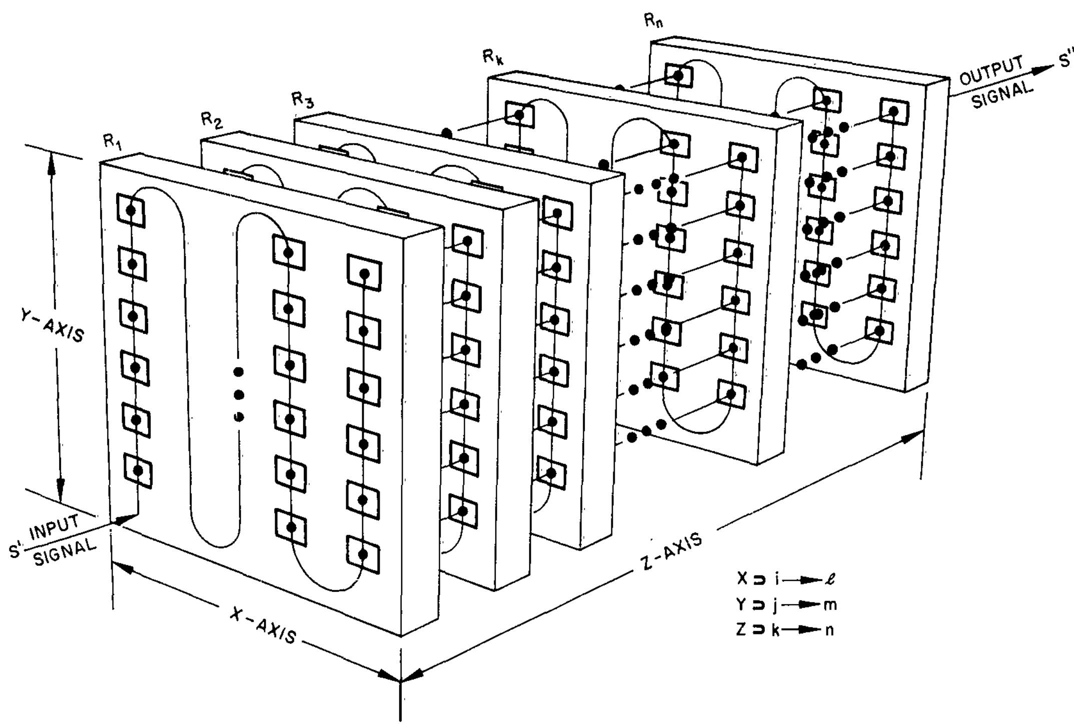

Fig. 3-2 Typical Three-Dimensional Matrix

$$R_t = \left[1 - (1 - r)^m\right]^n \tag{3.1}$$

can be converted readily into a three-dimensional matrix equation by changing the notation that indicates the total number of parallel components in a set (element). This change is required because the components in a set lie on two dimensions instead of one.

Thus, Eq. (3.1) becomes

$$R_t = \left[1 - (1 - r)^{\ell m}\right]^n \tag{3.2}$$

where

ℓ = number of components along the x-axis
m = number of components along the y-axis
n = number of components along the z-axis

By setting $\ell \cdot m$ equal to u , Eq. (3.2) becomes

$$R_t = \left[1 - (1 - r)^u\right]^n \tag{3.3}$$

Equation (3.3) describes the reliability of any three-dimensional matrix in which all components are equivalent and all elements are equivalent.

3.4 RELIABILITY OF A THREE-DIMENSIONAL MATRIX COMPOSED OF NONEQUIVALENT ELEMENTS

If the three-dimensional matrix is not a right prism (right parallelepiped), the elements will contain varying numbers of parallel components (u). The reliability equation can be generalized to cover these variations by accounting for each element individually. Thus, Eq. (3.3) becomes

$$R_t = \left\{\prod_{1}^{n}{}_k \left[1 - (1 - r)^{u_k}\right]_k\right\} \tag{3.4}$$

where k represents any given element from 1 to n.

Equation (3.4) can be expressed also in terms of component failure constants (λ) if

$$r_t \equiv r \quad \text{where} \quad r = \int_t^\infty \lambda e^{-\lambda t}\,dt \quad \text{or} \quad e^{-\lambda t}$$

Then, Eq. (3.4) becomes

$$R_t = \left\{ \prod_{1}^{n}{}_k \left[1 - \left(1 - e^{-\lambda t}\right)^{u_k} \right]_k \right\} \tag{3.5}$$

or

$$R_t = \left[1 - \left(1 - e^{-\lambda t}\right)^{u} \right]^n \tag{3.6}$$

where u_k is also equal to a constant.

Equations (3.4) and (3.5) describe the reliability of any three-dimensional matrix in which the reliability of any element may assume any value.

3.5 RELIABILITY OF A THREE-DIMENSIONAL MATRIX WHEN ALL COMPONENTS MUST FUNCTION

If it is assumed that every component in a matrix must function to achieve signal success – there are no failure factors in any term – the total reliability of any given element or slab can be expressed as follows:

$$R_{t_{s_k}} \equiv R_{s_k} = r^u$$

Then, Eq. (3.4) becomes

$$R_t = \left[\prod_{1}^{n}{}_k \left(r^{u_k} \right)_k \right] \tag{3.7}$$

or

$$R_t = \left[\left(r^u \right)^n \right] = \left[r^{nu} \right] \tag{3.8}$$

where u_k is equal to a constant.

Equations (3.7) and (3.8) can be expressed also in terms of component failure constants (λ) by setting r equal to $e^{-\lambda t}$. Equation (3.7) becomes

$$R_t = \left[\prod_{1}^{n}{}_k \left(e^{-\lambda t} \right)_k^{u_k} \right] \tag{3.9}$$

and Eq. (3.8) becomes

$$R_t = \left[\left(e^{-\lambda t} \right)^{nu} \right] = \left[e^{-nu\lambda t} \right] \tag{3.10}$$

Thus, Eqs. (3.7) and (3.9) describe the reliability of any three-dimensional matrix in which every component must function.

Section 4
APPLICATION OF THE BINOMIAL THEOREM TO MATRIX RELIABILITY FORMULAS

4.1 GENERAL

In Section 3, equations are developed for determining the reliability of any three-dimensional matrix in which the reliability of the individual series elements may assume any value. However, the elements in these equations are treated as single complex components with given reliability factors instead of component groups containing various combinations of failed and successful components. Since most practical matrix reliability problems are given in terms of the minimum number of successful components required or the maximum number of failed components allowable, the equations must be expanded to cover all possible combinations of failed and successful components in each element. This expansion is accomplished by applying the binomial theorem to the equations derived in Section 3.

4.2 CHANGE OF NOTATION

Before applying the binomial theorem to the matrix reliability equations, it is expedient to change the notation as follows:

Let

$$r = p$$

$$f = q$$

and

$$p + q = 1$$

Then, general Eq. (3.4) becomes

$$R_t = \prod_{1}^{n}{}_k \left[1 - (1 - p)^{u_k}\right]_k \tag{4.1}$$

where u_k varies with k and is equal to $\ell \cdot m$.

Or, since $p + q = 1$,

$$R_t = \prod_{1}^{n}{}_k \left[1 - q^{u_k}\right]_k \tag{4.2}$$

If u_k is a constant, Eq. (4.2) becomes

$$R_t = \left[1 - q^u\right]^n \tag{4.3}$$

or

$$R_t = \left[1 - C_u^u p^0 q^u\right]^n \tag{4.4}$$

and Eq. (3.8) becomes

$$R_t = \left[C_0^u p^u q^0\right]^n \tag{4.5}$$

$$= \left[1 \cdot p^u \cdot q^0\right]^n$$

$$R_t = \left[p^u\right]^n \tag{4.6}$$

4.3 APPLICATION OF THE BINOMIAL THEOREM

The term q^u from Eq. (4.3) is actually the last term in the binomial $(p+q)^u$. Therefore the expression $[1 - q^u]$ from Eq. (4.3) consists of all terms in the binomial expression $(p+q)^u$ except the last term.

Now, let (B. T.) represent any binomial term. Then, the expression $[1 - q^u]$ from

$$R_t = \left[1 - q^u\right]^n$$

may be written in binomial form as

$$R_t = \left[\sum_{1}^{(u+1)}{}_i \; (B.T.)_i - (B.T.)_{u+1}\right] \tag{4.7}$$

Equation (4.7) is equal to the sum of all the binomial terms of $(p+q)^u$ except the last term, $(B.T.)_{u+1}$ or q^u. Also, the value of p^u in Eq. (4.6) is really the first term in the binomial $(p+q)^u$.

4.4 REVERSING THE ORDER OF TERMS

In order to use the "Tables of the Cumulative" Binomial Probability Distribution,* the order of terms in the binomial series given in Eq. (4.7) must be changed as follows:

*The Staff of the Computation Laboratory, Tables of the Cumulative Binomial Probability Distribution, Vol XXXV, Cambridge, Mass., Harvard University Press, 1955

If the sum of all (B. T.)s is

$$\sum_{1}^{n+1} i\,(B.T.)_i$$

then the binomial series can be written as

$$\left[C_0^n\, p^0\, q^n + C_1^n\, p^1\, q^{n-1} + \ldots + C_r^n\, p^r\, q^{n-r} + \ldots + C_{n-1}^n\, p^{n-1}\, q^1 + C_n^n\, p^n\, q^0\right]$$

In this series, the normal order of terms in the binomial $(p + q)^n$ is reversed, i.e., complete failure is now represented by the <u>first</u> term $(C_0^n\, p^0\, q^n)$ and complete success is represented by the <u>last</u> term $(C_n^n\, p^n\, q^0)$.

This expression sums to

$$\sum_{i,\; S' = r_L = 0}^{S'' = r_U = n} \left[C_r^n\, p^r\, q^{(n-r)}\right]_i$$

If n is set equal to u the expression becomes

$$\sum_{i,\; S' = r_L = 0}^{S'' = r_U = u} \left[C_r^u\, p^r\, q^{(u-r)}\right]_i$$

In this form, the expression $[1 - q^u]$ consists of <u>all</u> the terms in the binomial $(p + q)^u$ except the <u>first</u> term; and the term p^u in Eq. (4.6) represents the last term in the binomial expression.

4.5 MATRIX RELIABILITY WHEN (u - r) COMPONENTS FAIL

If signal success occurs in the three-dimensional matrix except when at least the quantity (u - r) components fail in any element, the reliability of the matrix can be expressed as

$$R_t = \left[1 - \sum_{S'=r_L=0}^{S''=r_U=r} {}_i \left[C_r^u \, p^r \, q^{(u-r)} \right]_i \right]^n \tag{4.8}$$

or

$$R_t = \left[\sum_{S'=r_L=(r+1)}^{S''=r_U=u} {}_i \left[C_r^u \, p^r \, q^{(u-r)} \right]_i \right]^n \tag{4.9}$$

Then, for a complete general case where the elements have different u values, Eq. (4.8) becomes

$$R_t = \left\{ \prod_1^n {}_k \left[1 - \sum_{S'=r_L=0}^{S''=r_U=r} {}_i \left[C_r^{u_k} \, p^r \, q^{(u_k-r)} \right]_i \right]_k \right\} \tag{4.10}$$

and Eq. (4.9) becomes

$$R_t = \left\{ \prod_{1}^{n}{}_k \left[\sum_{S'=r_L=(r+1)}^{S''=r_U=u_k} {}_i \left[C_r^{u_k} p^r q^{(u_k-r)} \right]_i \right]_k \right\} \tag{4.11}$$

4.6 MATRIX RELIABILITY WHEN r COMPONENTS SUCCEED

If signal success occurs in the three-dimensional matrix <u>only</u> when at <u>least</u> r components <u>succeed</u> in any element, the reliability for the matrix can be expressed as

$$R_t = \left[1 - \sum_{S'=r_L=0}^{S''=r_U=(r-1)} {}_i \left[C_r^u p^r q^{(u-r)} \right]_i \right]^n \tag{4.12}$$

or

$$R_t = \left[\sum_{S'=r_L=r}^{S''=r_U=u} {}_i \left[C_r^u p^r q^{(u-r)} \right]_i \right]^n \tag{4.13}$$

For a complete general case where the elements have different u values, Eq. (4.12) becomes

$$R_t = \left\{ \prod_{1}^{n}{}_k \left[1 - \sum_{S'=r_L=0}^{S''=r_U=(r-1)} {}_i \left[C_r^{u_k} p^r q^{(u_k-r)} \right]_i \right]_k \right\} \tag{4.14}$$

and Eq. (4.13) becomes

$$R_t = \left\{ \prod_{1}^{n}{}_{k} \left[\sum_{S'=r_L=r}^{S''=r_U=u_k}{}_{i} \left[C_r^{u_k} p^r q^{(u_k-r)} \right]_i \right]_k \right\} \tag{4.15}$$